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TECHNICAL REPORT NO. 16

THE FAR FIELD ACOUSTIC WAVE PRODUCED IN N PARALLEL LIQUID LAYERS BY POINT SOURCE EXCITATION

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ABSTRACT

The problem of the far field acoustic wave produced in N lossless plane parallel liquid layers by point source excitation is solved in a form permitting of practical computation: Formal solutions of essentially the same problem, but not adaptable to convenient calculation, have been obtained at least as early as 1947 (Reference (b)).

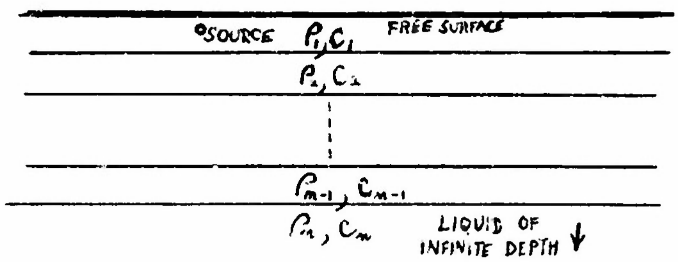
Pekeris has solved this problem in a computable form for the special cases of two and three layers (Reference (a)), terming the type of solution obtained "the normal mode form". The solution to be given here for the N layer case is similarly in the normal mode form, reducing to the results obtained by Pekeris for the two and three layer cases.

While the present solution is a generalization of the form obtained by Pekeris, the method of derivation is basically different. In particular, a fundamentally different method of obtaining the mode excitations is employed. Use of this method not only greatly simplifies finding the Green's function in the present problem, but it appears capable of facilitating the solution of various other excitation problems.

INTRODUCTION

The problem of the acoustic wave produced in N lossless plane parallel liquid layers by point source excitation has (in essence) been solved formally by Lurye in 1947 (reference (b)). However the first solutions of this problem which were in a form permitting of practical computation were obtained by Pekeris in 1948 for the special cases of two and three layers (reference (a)). We will obtain here, by basically different techniques, a generalization of Pekeris' form of solution to the N layer case.

Each of the parallel layers is, in itself, uniform and characterized by a density ρ_i and wave velocity c_i. What will be called the top layer is bounded by an acoustic free surface while the bottom layer is a liquid of infinite depth (forms a half space). We will consider the particular case where the source is in the top layer, the technique being essentially the same for the source in any layer. (In fact, if the solution is known for the source in the top layer, it can immediately be found for the source in any layer with the help of a simple reciprocity relation.) The system is shown in the accompanying figure.



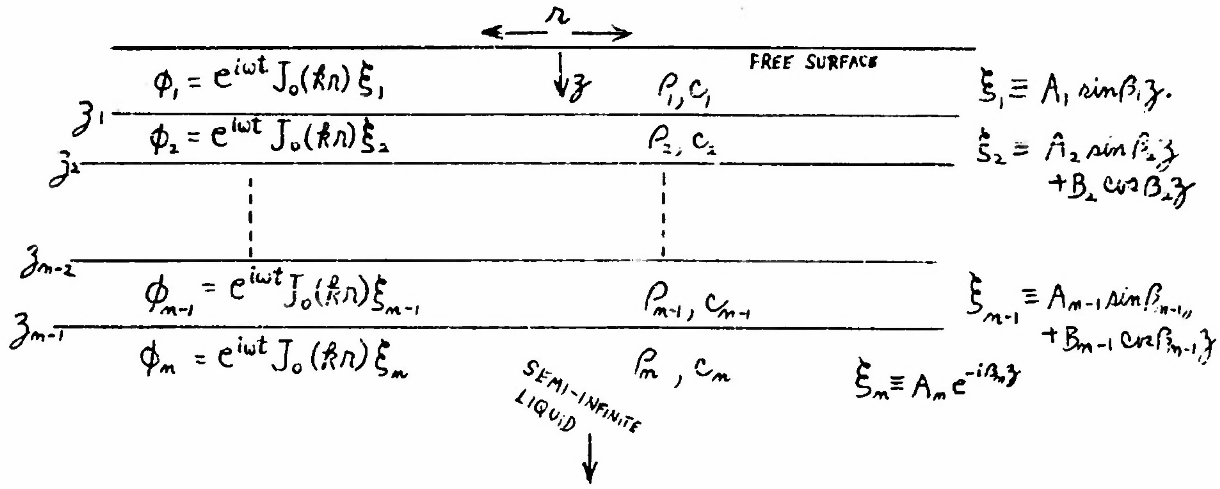
The solution will be obtained as a sum of normal modes. (By a mode is meant a fundamental solution of the wave equation which <u>by itself</u> satisfies all boundary conditions except those defining the source.) The significant fact about this form of solution is that it lends itself to practical numerical computation. In particular, no numerical integrations are involved. The lengthiest part of the computation, that of finding the phase velocities of the modes (i.e. - finding the eigenvalues), while often somewhat laborious, is still practical. As an examination of the "recurrence method" of Part I will show, the difficulty of finding the phase velocities does not in general increase very rapidly with the number of layers (a notable exception being the contrast between the two and three layer cases). Thus a six layer problem is not "tremendously more difficult" than a three layer problem.

After the phase velocities of the propagating modes are obtained, a method is used for finding the mode excitations which differs basically from the usual techniques. This approach greatly simplifies the present problem, and it is expected that it will be useful in many other excitation problems: For example, it should lend itself readily to solving the corresponding N layer problem for the case of elastic media. It has thus far been developed only to solve for the propagating modes (far field case). <u>In order to demonstrate the method most clearly, it will first be used to obtain the excitations for the simpler case of a layered rectangular wave guide (Part II)</u>, these results then being extended to point source excitation in plane parallel layers by an appropriate image technique (Part III). (If simplicity of illustration were not an important consideration, the technique of Part II, which is applied to a rectangular wave guide, could instead be applied directly to the parallel layer case, as described in paragraph 9c of Part II.)

The author wishes to thank Dr. Franklyn Levin for his careful reading of the manuscript and for his valuable suggestions.

Part I The Phase Velocities of the Modes

1. Referring to Fig. I, the velocity potential of a mode (solution of $\nabla^2\phi_j = \frac{1}{c_j^2}\frac{\partial^2\phi_j}{\partial t^2}$, where j denotes the layer) has the form in each layer as shown. It is seen, in order to satisfy the free surface boundary condition, that ξ_1 can contain no cosine term.



The boundary conditions are:

$$\rho_j \xi_j = \rho_{j+1}\xi_{j+1} \qquad \text{(Continuity of pressure)}$$

$$\frac{d}{dz}\xi_j = \frac{d}{dz}\xi_{j+1} \qquad \text{(Continuity of normal velocity)}$$

2. The β_j are given by:

$$\beta_j \equiv \sqrt{\frac{\omega^2}{c_j^2} - k^2} \equiv k\sqrt{\frac{c^2}{c_j^2} - 1} \quad (\text{For } j \neq m) \quad ; \quad \beta_m \equiv -k\sqrt{\frac{c^2}{c_j^2} - 1} \quad (\text{For } j = m)$$

Where C equals the phase velocity, which (like k) is the same in all layers. This relation is a direct consequence of the separation of variables in the wave equation of the particular layer, and independent of any boundary conditions. (Thus the same relation was used for the three layer case, as given in Eqns. (A141) of Reference (a), as for the present n layer case.) If we apply the two boundary conditions to surface Z_{m-1} (lowest surface) we obtain:

$$A_{m-1} = A_m e^{-i\beta_m z_{m-1}}\left[\frac{\rho_m}{\rho_{m-1}}\sin\beta_{m-1} z_{m-1} - i\frac{\beta_m}{\beta_{m-1}}\cos\beta_{m-1} z_{m-1}\right] \equiv A_{m-1}(k, A_m) \qquad \text{EQN (1)}$$

$$B_{m-1} = A_m e^{-i\beta_m z_{m-1}}\left[\frac{\rho_m}{\rho_{m-1}}\cos\beta_{m-1} z_{m-1} + i\frac{\beta_m}{\beta_{m-1}}\sin\beta_{m-1} z_{m-1}\right] \equiv B_{m-1}(k, A_m) \qquad \text{EQN (2)}$$

Applying the boundary conditions to any of the other surfaces (except the free surface) gives:

$$A_j = A_{j+1}\left[\frac{\rho_{j+1}}{\rho_j}\sin\beta_{j+1}z_j\sin\beta_j z_j + \frac{\beta_{j+1}}{\beta_j}\cos\beta_{j+1}z_j\cos\beta_j z_j\right] + B_{j+1}\left[\frac{\rho_{j+1}}{\rho_j}\cos\beta_{j+1}z_j\sin\beta_j z_j - \frac{\beta_{j+1}}{\beta_j}\sin\beta_{j+1}z_j\cos\beta_j z_j\right]$$

$$\equiv A_j(k, A_{j+1}, B_{j+1}) \qquad \text{EQN(3)}$$

$$B_j = A_{j+1}\left[\frac{\rho_{j+1}}{\rho_j}\sin\beta_{j+1}z_j\cos\beta_j z_j - \frac{\beta_{j+1}}{\beta_j}\cos\beta_{j+1}z_j\sin\beta_j z_j\right] + B_{j+1}\left[\frac{\rho_{j+1}}{\rho_j}\cos\beta_{j+1}z_j\cos\beta_j z_j + \frac{\beta_{j+1}}{\beta_j}\sin\beta_{j+1}z_j\sin\beta_j z_j\right]$$

$$\equiv B_j(k, A_{j+1}, B_{j+1}) \qquad \text{EQN (4)}$$

3. Thus, by applying Eqns. (1) and (2) to the lowest boundary, and (3) and (4) to successively higher boundaries till Z_1, we obtain the A's and B's in all layers as functions of k and A_m. But another requirement that k correspond to a mode is that $B_1(k, A_m) = 0$. (Since in the top layer ξ_1 contains no cosine term.) The k values for which $B_1 = 0$ are in general best found by graphing B_1 against k (the A_m value has no effect upon the roots). (Graphing proves to be the practical procedure even for the three layer case.) It is seen that only real values of k which satisfy the condition correspond to propagating modes. (Note also that β_m must be negative and imaginary to represent a propagating mode.)

4. Summary of Part I

a) Results Obtained

The k's of the propagating modes and the linear relations between the A's and B's of each mode have been found. The remaining basic problem (in connection with solving for the far field) is to find the point source excitations of these modes.

b) Essential Principle Employed

Starting at the lowest surface and applying the boundary conditions in turn to successively higher surfaces, the fundamental solution in each layer is obtained in terms of the solution in the lowest medium (i.e. - in terms of any k and any A_m without any of the other A's and B's). But the only way the top layer solution can be thus related to the bottom solution and at the same time satisfy the free surface boundary condition is by permitting only certain discrete k values. (That is the k values which satisfy all these boundary conditions, and therefore correspond to modes, are simply the values which satisfy

the <u>single</u> equation $B_1(k, A_m) = 0$. While only a single equation need be satisfied, and <u>therefore solution by graphing is generally feasible</u>, the equation of course does become more complicated as the number of layers increases.)

5. <u>Additional Comments</u>

a) A necessary condition for the existence of propagating modes is that the wave velocities C_j must <u>not</u> be monatonically increasing with successively higher layers. (Or else there would be no total reflection mechanism to keep energy from escaping through the bottom.)

b) In general some of the layers may have imaginary β's corresponding to propagating modes (except that β_m must always be negative and imaginary). The mathematics above is formally the same whether or not certain of the β's are imaginary (i.e. - in a layer where β is imaginary the sine and cosine terms become hyperbolic).

c) It is interesting to compare the A_j and B_j coefficients above with the A, B, C, D quantities in Eqns. (A8), (A9), (A10) of Reference (a). Though there is a superficial resemblence, it will be seen that these two sets of quantities are basically different.

Part II Far Field Due to a Point Source in Layered Rectangular Guide

1. It will be useful first to consider the problem of point source mode excitation in the rectangular wave guide case. Not only will the result prove convenient for solving the corresponding parallel layer problem, but this simpler case will also serve to more clearly illustrate a fundamentally different technique for finding Green's functions.

2. Consider the uniform rectangular wave guide shown in cross section in Fig. II: Let the width, w, of the guide be too small to permit any propagating modes other than those which are independent of y to exist (for the fixed frequency ω which we will consider). Assume further a large number, s, of point sources all at a depth, d, to exist as shown in Fig. III. Let all be oscillating at the same frequency ω, all be of volume displacement δV (source strength $=\delta V$), and of the same spatial phase relationships as an m'th propagating mode wave traveling in the $+x$ direction. Thus:

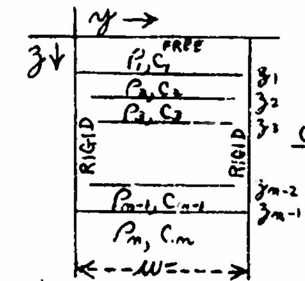


FIG II
CROSS SECTION

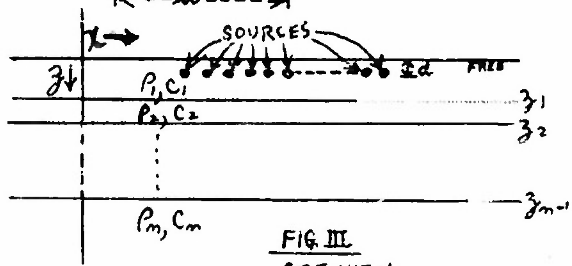


FIG III
SIDE VIEW

$$\left[\text{PHASE DIFFERENCE BETWEEN ANY TWO SOURCES}\right] = 2\pi\left[\frac{\text{DIFFERENCE IN THEIR } x \text{ COORDINATES}}{\text{GUIDE WAVE LENGTH OF } m\text{'th MODE}}\right]$$

Then for s sufficiently large, we conclude from simple phase considerations, that the far field radiation produced by these sources consists of the m'th mode traveling in the $+x$ direction with all other modes negligibly small in comparison; the m'th mode wave traveling in the $-x$ direction is also seen to be negligibly small.

(Note: (a) Strictly speaking, to ensure that all other propagating modes be small, we may stipulate that the sources be spaced uniformly at a distance apart less than the shortest possible propagating guide wave length for frequency ω.

(b) Actually the derivation to follow could have been accomplished even if only one source existed, so that the other modes would not be negligibly small; but the present method is simpler.)

3. Calling the velocity potential of this radiated m'th mode Φ_I, we assume in addition that another wave Φ_{II}, also of the m'th mode and traveling in the $+x$ direction, has been introduced by sources at $x=-\infty$. (Φ's will be used for the layered rectangular wave guide to distinguish from the ϕ's used for the horizontally unbounded parallel layer case.) Further assume that

Φ_{II} is a much stronger signal than Φ_I: Thus if $\eta_{II}(z)$ is "the z part" of the Φ_{II} solution (that is $\Phi_{II} = \eta_{II}(z) \sin(kx - \omega t + \theta_0)$); and $\eta_I(z)$ is "the z part" of Φ_I, we have $|\eta_{II}| \gg |\eta_I|$. Also let the sources producing the Φ_I wave be constrained to oscillate as previously described; these oscillations being unaffected by the strong Φ_{II} signal passing by. (The word "sources" will always refer to those producing the Φ_I wave.) Assume finally that the Φ_{II} wave is 180° out of phase with the Φ_I wave. Then in regions of large $+x$ the field is given by $(|\eta_{II}| - |\eta_I|) \sin(kx - \omega t + \theta_0)$.

Therefore since the field approaching the source is given by amplitude η_{II}, and the (far) field traveling away is given by the slightly smaller amplitude $(|\eta_{II}| - |\eta_I|)$, it follows from conservation of energy that the sources are absorbing power (rather than losing it). Therefore we may write:

$$\left[\text{NET POWER RECEIVED BY SOURCES}\right] = \left[\text{POWER OF } \eta_{II} \text{ FIELD}\right] - \left[\text{POWER OF } (|\eta_{II}| - |\eta_I|) \text{ FIELD}\right] \qquad \text{EQN(5)}$$

Where by "η_{II} field" is simply meant the "Φ_{II} field"; the "η field" terminology being introduced for conciseness, since the immediately following steps involve the η's rather than the Φ's. We next introduce the evident relation:

$$\frac{\left[\text{POWER OF } (|\eta_{II}| - |\eta_I| \text{ FIELD})\right]}{\left[\text{POWER OF } \eta_{II} \text{ FIELD}\right]} = \frac{\left[|\eta_{II}(a)| - |\eta_I(a)|\right]^2}{\left[\eta_{II}(a)\right]^2}$$

Where the values of $\eta_{II}(z)$ and $\eta_I(z)$ have, merely for concreteness, been taken at some arbitrary fixed depth $z = a$. Putting this relation into Eqn. (5) gives:

$$\left[\text{NET PWR RECEIVED BY SOURCES}\right] = \left[\text{PWR OF } \eta_{II} \text{ FIELD}\right] \left(1 - \frac{\left[|\eta_{II}(a)| - |\eta_I(a)|\right]^2}{\left[\eta_{II}(a)\right]^2}\right)$$

Since $\eta_{II}(a) \gg \eta_I(a)$, this may be written:

$$\left[\text{NET PWR RECEIVED BY SOURCES}\right] = \left[\text{PWR OF } \eta_{II} \text{ FIELD}\right] \left(\frac{2|\eta_I(a)|}{\eta_{II}(a)}\right) \qquad \text{EQN (5A)}$$

4. Now since the self produced pressure at a mathematical point source is finite and continuous (unlike the particle velocity), and since $|\eta_{II}| \gg |\eta_I|$, it follows that the pressure produced at any of the sources due to the other sources, or the self produced pressure, is small compared to that of the η_{II} field. Further the η_{II} and η_I waves, being 180° out of phase, produce the weakest possible resultant field; so that from Eqn.(5) we conclude that the phasing of η_{II} is such that the sources are absorbing the maximum power possible: Therefore since the pressure at the sources is essentially that of the η_{II} field, and this field is phased to do a maximum of work on the sources (the net local pressure field works against the "pulsating volume displacement" of the source), we may write:

$$\left[\text{NET POWER GIVEN TO THE } s \text{ SOURCES}\right] = \frac{s}{(1 \text{ CYCLE})} \int_{1 \text{ CYCLE}} (P_s \cos\omega t)\left(\frac{d}{dt}\,\delta V \sin\omega t\right) dt$$

Where P_s is the maximum magnitude of pressure of the η_{II} field occurring at the sources, and δV is the volume displacement of each source. Hence Eqn. (5A) may now be written:

$$\frac{|\eta_I(a)|}{|\eta_{II}(a)|} = \frac{(\text{PWR GIVEN SOURCES})}{2(\text{PWR OF } \eta_{II} \text{ FIELD})} = \frac{s\int_{1\text{CYCLE}}(P_s\cos\omega t)(\frac{d}{dt}\,\delta V \sin\omega t)\,dt}{2\int_{1 \text{ CYCLE}}\left\{\underbrace{\int p v_x \, dS}_{\text{OVER COMPLETE CROSS SECTION OF GUIDE}}\right\}dt} \equiv \frac{s P_s \delta V \int_{\omega t=0}^{2\pi} \cos^2\omega t \, d\omega t}{2\omega \int_{\text{CYCLE}}\left\{\int_{z=0} p v_x \, dz\right\}dt}$$

$$\frac{|\eta_I(a)|}{|\eta_{II}(a)|} = \frac{s\pi P_s \delta V}{2\omega\int_{\text{CYCLE}}\left\{\int_{z=0}^{\infty} p v_x \, dz\right\}dt} \qquad \text{EQN. (5B)}$$

Where p and v_x are the instantaneous real values of η_{II} field pressure and x component of particle velocity at any point.

5. We have $\Phi_{II} = \eta_{II}(z)\sin(kx - \omega t + \theta_0)$. Therefore since $v_x = -\frac{\partial \Phi_{II}}{\partial x}$ and $p = \rho_j \frac{\partial \Phi_{II}}{\partial t}$ (where ρ_j = density in j th rectangular wave guide layer). we have:

$$\int_{\text{CYCLE}}\left\{\int_{z=0}^{\infty} p v_x \, dz\right\}dt \equiv \int_{\omega t=0}^{2\pi}\left\{\sum_{j=1}^{n}\rho_j\int_{z_{j-1}}^{z_j}\left(-\eta_{II_j}\omega\cos(kx-\omega t+\theta_0)\right)\left(-\eta_{II_j} k\cos(kx-\omega t+\theta_0)\right)dz\right\}dt$$

$$\equiv k\omega\sum_{j=1}^{n}\rho_j\int_{z_{j-1}}^{z_j}\eta_{II_j}^2\left[\int_{\omega t=0}^{2\pi}\cos^2(kx-\omega t+\theta_0)\,dt\right]dz$$

Hence

$$\left[\text{PWR OF } \eta_{II} \text{ FIELD}\right] = \omega\int_{\text{CYCLE}}\left\{\int_{z=0}^{\infty} p v_x \, dz\right\}dt = k\pi\omega\sum_{j=1}^{n}\rho_j\int_{z_{j-1}}^{z_j}\eta_{II_j}^2\,dz \qquad \text{EQN (6)}$$

Where η_{II_j} equals η_{II} in the j'th layer; the j subscript merely emphasizing that η_{II} is given by different expressions in the various layers.

Also, we have:

$$P_s = \rho_1\left(\frac{\partial\Phi_{II}(x,z=d,t)}{\partial t}\right)_{MAX} \equiv \rho_1\left|\eta_{II1}(d)\right|\left(\frac{\partial}{\partial t}\sin(kx-\omega t+\theta_0)\right)_{MAX} \equiv \rho_1\omega\left|\eta_{II1}(d)\right|$$

Substituting from this relation and from Eqn. (6) into Eqn. (5B) gives:

$$\frac{|\eta_I(a)|}{|\eta_{II}(a)|} = \frac{s\rho_1\omega\,\delta V\,|\eta_{II1}(d)|}{2k\omega\sum_{j=1}^{m}\rho_j\int_{z_{j-1}}^{z_j}\eta_{IIj}^2\,dz} \qquad \text{EQN. (5C)}$$

6. If we choose $a = d$, then

$$\eta_I(a) \equiv \eta_I(d) \equiv a_{I1}\sin\beta_1 d$$

Where a_{I1} is the coefficient "designating the mode strength" (in the top layer) of the η_I wave. It is important to notice that the set of β's (and k's) which correspond to modes in the present rectangular wave guide case are identical to those applying in the corresponding parallel layer problem of Part I. Further a set of coefficients a_j and B_j apply in the present case which have precisely the same linear relationships as the corresponding A_j and B_j of Part I. This similarity occurs as a result of the fact that the "z part of the solution" is the same (within a multiplicative constant) in both the parallel layer and rectangular wave guide cases.
For $a = d$, Eqn. (5C) may therefore be written:

$$\left|a_{I1}\sin\beta_1 d\right| = \frac{(\eta_{II1}(d))^2 s\rho_1\omega\,\delta V}{2k\omega\sum_{j=1}^{m}\rho_j\int_{z_{j-1}}^{z_j}\eta_{IIj}^2\,dz} \qquad \text{EQN (5D)}$$

Since all the sources reinforce for the m'th mode in question, we may conclude from the linear superposition properties of amplitudes, that the corresponding expression for the a_1 coefficient for a single point source is:

$$\boxed{\left|a_{j=1\text{ LAYER};\,m'\text{th MODE}}\right| \equiv \left|a_{1m}\right| = \frac{\rho_1\omega(\eta_{II1m}(d))^2\Delta V}{2k_m\omega\sin\beta_{1m}d\sum_{j=1}^{m}\rho_j\int_{z_{j-1}}^{z_j}\eta_{IIjm}^2\,dz} \qquad \text{EQN (5E)}}$$

Where ΔV is the volume displacement of the source. $\eta_{\text{II}jm}$ represents not the m'th mode produced by the source, but a "mathematically fictional" m'th mode of arbitrary strength. $\eta_{\text{II}1m}(d)$ is the corresponding "fictional mode" value at $z = d =$ (depth of source). $\eta_{\text{II}jm}$ constitutes a dummy variable which drops out (essentially cancels the $\eta_{\text{II}m}(d)$) when the evaluation of $|a_{1m}|$ is carried through.

Thus Eqn. (5E) determines the excitation of any propagating mode, after the method of Part I is used to get the β_{km} and the linear relations between the a_{jm} and b_{jm}, for the case of a point source in the top layer of an n layer rectangular guide. (Remembering that the β_{jm} are identical with those of the corresponding parallel layer problem of Part I, and that the set of a_{jm} and b_{jm} coefficients have the same linear relations as the A_{jm} and B_{jm} of Part I.)

7. While the linear relations between the a_{jm} and b_{jm} coefficients are obtainable by the method of Part I, and the magnitude of a_{1m} is determined by Eqn. (5E), the problem of the relative phasing (temporal) between the different modes at any value of x has yet to be solved: To do this, we observe that the fictitious $\eta_{\text{II}jm}$ fields, corresponding to the various propagating modes of a fixed frequency ω, all bear the same phase relationship to the source (since all do maximum work on it). That is all the $\eta_{\text{II}jm}$ modes are in phase with each other at the source. But since all the radiated propagating modes are in opposite phase to their corresponding fictitious $\eta_{\text{II}jm}$ fields, these radiated modes must also all be in phase at the source. Therefore we have for any propagating mode:

$$\Phi_{\text{I}1m;\ \text{FOR } z \text{ CONSTANT}} \sim \sin(k_m x - \omega t + \theta_c).$$

Where θ_c is a constant which describes the phasing of the source. Thus we may write the solution:

$$\Phi_{\text{I}1m} \equiv \Phi_{1m} = \eta_{1m} \sin(k_m x - \omega t + \theta_c) \equiv a_{1m} \sin \beta_{1m} z \cdot \sin(k_m x - \omega t + \theta_c) \qquad \text{EQN (7)}$$

The phase angle of Φ_{1m} may be completely accounted for by the $\sin(k_m x - \omega t + \theta_c)$ factor; so that the coefficient a_{1m} may be taken as real. And from the relations given in Part I, a_{1m} being real causes all the other a_{jm} and b_{jm} to be real.

8. Summary of Part II

a) Results obtained: Solved the far field problem for point source excitation of the n layered rectangular wave guide shown in Fig. II. More specifically, we

have obtained the a_{lm} coefficient; so that all the a_{lm} and B_{lm} may be obtained by the method of Part I. Also, the relative phasing of the different propagating modes has been obtained.

b) Essential Principles Employed in the Analysis: The net energy radiated or absorbed by a source is seen to be the work done by or on the source as it "pulsates against the surrounding pressure field." Now, while the work of the source against its self-produced pressure field cannot be calculated (the field produced by the source being precisely the unknown we wish to find), the work done when a very strong and known externally introduced signal travels past the source can to good accuracy be calculated: The unknown self-produced field of the source contributing a negligible fraction of the work in the latter case. (To understand this we note that the self-produced pressure at the point source is finite, unlike the particle velocity.) If the pulsating source is phased so as to absorb energy from the oncoming wave then, by conservation of energy, the field radiated by the source must combine with the externally produced wave so that the power of the net wave traveling away is less than that of the original incident signal. Thus by knowing the work done on the pulsating source by a suitable strong field passing over it, the field radiated by the source is determined. A more precise description of the principles involved was given in paragraph 3.

9. Additional Comments

a) While the case treated was for the source in the top layer of the wave guide, it is seen that the same technique could be applied for the source in any layer.

b) It is seen that only the propagating modes are obtained by the method per se. That is, only the far field solution is obtained. No attempt has as yet been made to obtain either the non-propagating mode excitations or what corresponds to the branch line integral appearing in Eqn. (A46) of reference (a).

c) The technique employed here could have been applied directly to the parallel layer case by considering the effect of an externally produced cylindrical mode in the form of an incoming wave as it "collapses on a source" which is constrained to oscillate in a prescribed manner. However, since clarity of illustration is important, the method was applied to the considerably simpler rectangular wave guide case. (For example, the rectangular wave guide case was simpler in that it was easier to show that each mode can be "considered independently" from the others.)

Part III Far Field in n Parallel Liquid Layers Due to Point Source Excitation.

1. In Part II we found the point source excitations of the propagating modes for the case of layered rectangular guide. This result will now be extended to the case of point source excitation in n parallel liquid layers. To relate these two cases we first consider the following image equality:

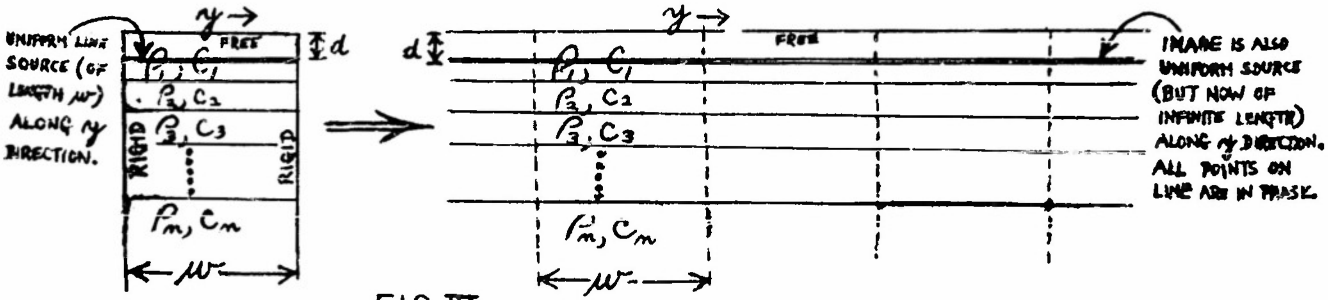


FIG IV

This image equality states that the field seen in the rectangular wave guide due to the line segment source is identical to the field seen in a parallel layer region due to an infinite line source, as shown.

2. The fundamental m'th mode solution of the wave equation in cylindrical coordinates may, for the top layer, be expressed in the real form (which will be somewhat more convenient in what follows):

$$\phi_{1m} = A_{1m} J_0(k_m r)\left(\sin\beta_{1m} z\right)\cos(\omega t + \epsilon_m)$$

Here A_{1m} is taken as real, the phase angle ϵ_m "absorbing any phase angle that A_{1m} may otherwise have". (Ultimately, ϵ_m will depend upon the phasing of a point source with respect to the time t.) A consequence of A_{1m} being real is that all the A_{jm} and B_{jm} of the mode will be real: That is, all the ϕ_{jm} of the mode are real. The subscript on ϵ_m indicates the possibility that ϵ_m depends upon the mode. (But it will turn out to be independent of mode.) It is seen that the above expression for ϕ_{1m} represents a standing wave. Using the asymptotic form of $J_0(k_m r)$ for r large, the corresponding far field solution becomes:

$$\phi_{\substack{1m \\ r\to\infty}} = A_{1m} \sin\beta_{1m} z \cdot \left[\sqrt{\frac{2}{\pi k_m r}}\cos\left(k_m r - \frac{\pi}{4}\right)\right]\cos(\omega t + \epsilon_m)$$

The outgoing wave part of the far field standing wave is:

$$\phi_{\substack{1m\,(\text{OUTGOING}) \\ r\to\infty}} = A_{1m} \sin\beta_{1m} z \cdot \left(\frac{1}{\sqrt{2\pi k_m r}}\right)\cos\left[\left(k_m r - \frac{\pi}{4}\right) - \omega t - \epsilon_m\right] \quad \text{EQN (8)}$$

It is this outgoing wave part that corresponds to radiation from a point source, the incoming wave part being discarded (as is usually done with the "physically unreasonable" advanced potential). (Note that A_{1m} and ϕ_{m} are the symbols used in the cylindrical coordinate case as against the corresponding a_{1m} and Φ_{1m} of the rectangular case of Part II.) Referring now to the equivalent parallel layer image of Fig. IV, the net field produced (by all the modes) at a <u>large distance b</u> from the line source, and in the top layer, is:

$$\left[\text{TOTAL VELOCITY POTENTIAL}\right] \equiv \mathcal{J} = \sum_{\text{ALL MODES}} \left(\frac{dA_{1m}}{dy}\right) \frac{\sin \beta_{1m} z}{\sqrt{2\pi k_m}} \int_{y=-\infty}^{\infty} \frac{\cos\left[\left(k_m\sqrt{b^2+y^2} - \frac{\pi}{4}\right) - \omega t - \epsilon_m\right]}{\sqrt[4]{b^2+y^2}}\, dy$$

Where $\left(\frac{dA_{1m}}{dy}\right)$ is a constant for a given mode and is analagous to the A_{1m} coefficient of the "<u>cylindrical</u>" case. More precisely the effective A_{1m} coefficient of a short element δy of the line source (δy acts essentially like a point source) would have the value $\left(\frac{dA_{1m}}{dy}\delta y\right)$.

3. From the image equality of Fig. IV the quantity $\mathcal{J}$ of Eqn. (9) can be seen to represent the total velocity potential inside of a rectangular wave guide (due to a line segment source): This physical fact can lead to evaluating the integral in Eqn. (9). (Further, as is often the case in optics, this integral has the simplifying property of being evaluable over a stationary phase region given by $|y| \ll b$. More precisely, calling the "contributing range of y" $\equiv y_{STAT.}$, we have as $b \to \infty$ that $y_{STAT.} \to \infty$, but so that $\frac{y_{STAT.}}{b} \to 0$.)

However, the integral has essentially been evaluated by a relation given on pg. 416 of Watson's "Bessel Functions", 2nd Edition. The equation is:

$$\int_0^{\infty} \frac{J_0(kR)\, e^{-a\sqrt{R^2-b^2}}}{\sqrt{R^2-b^2}} R \cdot dR = \frac{e^{ib\sqrt{a^2+k^2}}}{\sqrt{a^2+k^2}}$$

If we put $a=0$, the integrand becomes imaginary for $R<b$, and real for $R>b$; so that for the $a=0$ case the real part of this equation becomes:

$$\int_b^{\infty} \frac{J_0(kR)\, R\, dR}{\sqrt{R^2-b^2}} = \frac{\cos kb}{k}$$

For b large this may be written:

$$\sqrt{\frac{2}{\pi k}}\int_{b}\frac{\cos(kR-\frac{\pi}{4})\cdot\sqrt{R}\,dR}{\sqrt{R^2-b^2}} = \frac{\cos kb}{k}$$

If we put $R = \sqrt{b^2+y^2}$, this becomes:

$$\sqrt{\frac{2}{\pi k}}\int_{y=0}^{\infty}\frac{\cos(k\sqrt{b^2+y^2}-\frac{\pi}{4})\,dy}{\sqrt[4]{b^2+y^2}} = \frac{\cos kb}{k}.$$

If this last equation is expressed in complex form and multiplied by $e^{-i(\omega t+\epsilon_m)}$, the real part of the resulting equation is:

$$\sqrt{\frac{2}{\pi k}}\int_{y=0}^{\infty}\frac{\cos(k\sqrt{b^2+y^2}-\frac{\pi}{4}-\omega t-\epsilon_m)}{\sqrt[4]{b^2+y^2}}\,dy = \frac{\cos(kb-\omega t-\epsilon_m)}{k} \qquad \text{EQN (10)}$$

Identifying this integral with that appearing in Eqn. (9), gives for $\mathcal{J}$:

$$\mathcal{J} = \sum_{\text{ALL MODES}}\left(\frac{dA_{1m}}{dy}\right)\frac{\sin\beta_{m}z}{k_m}\cdot\cos(k_m b-\omega t-\epsilon_m)$$

4. Since $\mathcal{J}$ is also the field seen in the <u>rectangular wave guide</u> (by the image equality of Fig. IV); and since the <u>summand</u> is of the form $[\text{CONSTANT}]\cdot\cos(k_m b-\omega t-\epsilon_m)$, the summand is recognized as constituting the <u>m'th rectangular wave guide mode</u>. (i.e., The summand is an undamped sinusoidal function of $k_m b$, as is the case for propagating rectangular wave guide modes.) Thus we may write:

$$\left[\begin{array}{l}\text{RECTANGULAR WAVE GUIDE VELOCITY POTENTIAL}\\ \text{OF } m\text{'th MODE AT } x=b \text{ AND IN TOP LAYER}\end{array}\right] \equiv \mathcal{J}_m = \left(\frac{dA_{1m}}{dy}\right)\left(\frac{1}{k_m}\right)\sin\beta_{1m}z\cdot\cos(k_m b-\omega t-\epsilon_m)$$

Next we observe that $\mathcal{J}_m$ may be regarded as the rectangular wave guide m'th mode solution for a <u>point source</u> of a volume displacement ΔV equal to that of the segment source of Fig. IV (of length w). (The far field equivalence for a rectangular wave guide between the line segment source and a point source of equal displacement can be understood from the fact that the same work is done on either type source by the "hypothetical $\Phi_{II,m}$ field" of Part II, provided only that w is small enough so that the propagating mode amplitudes are independent

of y. (The value of w in no way affects the equivalent parallel layer image.)) Therefore we have that $\mathcal{J}_m$ may also be expressed by Eqn. (7):

$$\mathcal{J}_m = (\Phi_{1m})_{x=\ell} = a_{1m} \sin\beta_1 z \cdot \sin(k_m x - \omega t + K)$$

Where a_{1m} is given by Eqn. (5E) when ΔV in that equation is put equal to $w\left(\frac{dV}{dy}\right)$, where $\left(\frac{dV}{dy}\right)$ is the strength per unit length of the line source; and where K is some phase angle indicating the fact that the phasing of the line source has not been defined. Comparing the coefficients of these last two expressions for $\mathcal{J}_m$ gives:

$$a_{1m} = \left(\frac{dA_{1m}}{dy}\right)\left(\frac{1}{k_m}\right) \qquad \text{EQN (11)}$$

Eqn. (11) relates the unknown $\left(\frac{dA_{1m}}{dy}\right)$ of the cylindrical case to the known a_{1m} of the rectangular wave guide case. It leads to a connection of the point source mode excitation in the cylindrical case to the previously determined excitation in the rectangular guide case.

5. Eqn. (11) may be written:

$$\frac{dA_{1m}}{dy} \equiv \frac{dA_{1m}}{dV}\frac{dV}{dy} = k_m a_{1m}$$

Where $\left(\frac{dV}{dy}\right)$ is the volume displacement per unit length of the line source of Fig. IV (as already defined). Dividing through by $\left(\frac{dV}{dy}\right)$ gives:

$$\frac{k_m a_{1m}}{\left(\frac{dV}{dy}\right)} = \frac{\frac{dA_{1m}}{dy}\cdot\delta y}{\frac{dV}{dy}\cdot\delta y} \equiv \frac{\delta A_{1m}}{\delta V}$$

Where δA_{1m} is the effective A_{1m} coefficient (cylindrical case) due to radiation from the differential line source element δy. And where δV is the total volume displacement of the element δy. Identifying the differential element δy with an effective point source, and calling $\delta A_{1m} \equiv A_{1m}$ (to correspond to the nomenclature for a point source in the cylindrical case), the last equation may be written:

$$A_{1m} = \frac{k_m a_{1m}}{\left(\frac{dV}{dy}\right)}\delta V \qquad \text{EQN (11A)}$$

To obtain $|a_{1m}|$ (a_{1m} is real) we replace ΔV by $w\left(\frac{dV}{dy}\right)$ in Eqn. (5E):

$$|a_{1m}| = \frac{\rho_1 \omega \left(\eta_{\text{II}1m}(d)\right)^2 \cdot w\left(\frac{dV}{dy}\right)}{2 k_m w \left|\sin\beta_{1m} d\right| \sum_{j=1}^{n} \rho_j \int_{z_{j-1}}^{z_j} \eta_{\text{II}jm}^2 \, dz}$$

Putting this value for $|a_{1m}|$ into Eqn. (11A) gives:

$$|A_{1m}| = \frac{\rho_1 \omega \cdot \left(\eta_{\text{II}1m}(d)\right)^2 \cdot \delta V}{2 \left|\sin\beta_{1m} d\right| \sum_{j=1}^{n} \rho_j \int_{z_{j-1}}^{z_j} \eta_{\text{II}jm}^2 \, dz} \qquad \text{EQN (11B)}$$

Eqn. (11B) gives the magnitude of A_{1m} for a point source of volume displacement δV and at a depth d in the top layer in the parallel layer (i.e. - cylindrical) case. The other quantities appearing in Eqn. (11B) are identical with those defined in connection with Eqn. (5E).

6. The temporal phasing of the modes due to point source excitation in the parallel layer case is yet to be determined:

   Comparing the integrand to the value of the integral in Eqn. (10), we see that the signal from a line source a distance b away (corresponds to integral) lags the signal from a point source the same distance away (corresponds to the integrand when $y=0$) by $\frac{\pi}{4}$ in any given mode. And by the image equality of Fig. IV the mode phasing from the infinite line source is the same as for the rectangular wave guide case. Thus the far field m'th mode signal due to a point source in the parallel layer case is of phase angle $\frac{\pi}{4}$ less than the corresponding (same distance from similarly phased source) signal in the rectangular guide case.

7. Let us assume that the point source is phased in time so as to have its instantaneous volume displacement varying as $\cos\omega t$. First we consider the rectangular wave guide case for this source: Referring to Part II, since the $\eta_{\text{II}jm}$ field does a maximum of work on the source, its pressure can be seen to lag the instantaneous source displacement by 90°. Therefore the radiated mode, being out of phase with the $\eta_{\text{II}jm}$ field, must have its pressure leading the source displacement by 90°. And since $p = \rho \frac{\partial \phi}{\partial t}$, it readily follows that the instantaneous source displacement

is in phase with the velocity potential (at the source) of any of the radiating modes. Hence the temporal phase of the velocity potential of a mode due to the source may be expressed at a point (x, z) in the top layer by:

$$\Phi_{m\,(\text{FOR } z=\text{CONST})} \sim \pm\cos(k_m x-\omega t) \equiv \pm\cos(\omega t-k_m x)$$

The $\pm$ factor indicates that an additional 180° phase shift between observer and source must be introduced when their respective depths are such that $\sin\beta_{1m}z$ and $\sin\beta_{1m}d$ are opposite in sign. (Since the temporal phasing of a given mode alternates with depth as the sign of $\sin\beta_{1m}z$.) Further, as is required, the argument of the cosine is such that if the observer is at the source $(x=0, z=d)$ the expression becomes identical to that for the phase of the instantaneous source displacement:

$$\Phi_m(0,d,t) \sim +\cos(-\omega t) \equiv +\cos(+\omega t).$$

From paragraph 6 it follows we may obtain the corresponding parallel layer phasing (far field) simply by subtracting $\frac{\pi}{4}$ from the phase of Φ_m (and replacing the x coordinate by r):

$$\phi_{m\,(\text{FOR } z=\text{CONST})} \sim \pm\cos(\omega t-k_m r-\tfrac{\pi}{4})$$

8. Thus we know the phase of a far field radiated mode in the parallel layer case, and from Eqn. (11B) we know the magnitude of the corresponding A_{1m} coefficient. These results may be expressed more conveniently if we generalize Eqn. (11B) by defining A_{1m} as a (real) quantity whose sign depends upon the depth of the source:

$$\boxed{A_{1m} = \frac{\rho_1\omega\,(\eta_{\text{II}1m}(d))^2 \cdot \delta V}{2\sin\beta_{1m}d\,\sum\limits_{j=1}^{n}\rho_j\int_{z_{j-1}}^{z_j}\eta_{\text{II}jm}^2\,dz}} \qquad \text{EQN (11C)}$$

Where this value of A_{1m} is to be inserted into the following solution for the far field radiated mode in the top layer:

$$\boxed{\phi_{1m\,(\text{OUTGOING})\,r\to\infty} = A_{1m}\sin\beta_{1m}z\left(\frac{1}{\sqrt{2\pi k_m r}}\right)\cos(\omega t-k_m r-\tfrac{\pi}{4})} \qquad \text{EQN (12)}$$

The form of this last equation can be seen to follow directly from the corresponding expression in paragraph 2.

In general the solution in any of the layers is given by:

$$\boxed{\phi_{\substack{jm\,(\text{OUTGOING})\\ r\to\infty}} = \xi_{jm}\left(\frac{1}{\sqrt{2\pi k_m r}}\right)\cos\left(\omega t - k_m r - \frac{\pi}{4}\right)} \qquad \text{EQN (13)}$$

Where the ξ_{jm} are defined in Fig. I (in Part I). (The m subscripts, representing the mode, do not appear in Fig. I.)

9. <u>Summary of Part III</u>

a) <u>Results Obtained</u>: Solved the far field problem for point source excitation for the case of n parallel liquid layers. More specifically, we have obtained the A_{1m} coefficient (defined in Fig. I) for the case of a point source; so that all the A_{jm} and B_{jm} may be obtained by the method of Part I. Also the phasing of the propagating modes with respect to the phasing of the source have been obtained.

10 b) <u>Essential Principles Employed</u>: It will be simpler if we describe a slight variation of the method employed. (Though the method employed was more convenient for the actual derivation.) Understanding the variation will basically amount to understanding the original method. If in Fig. IV the line source segment in the rectangular wave guide is replaced by a point source midway between the guide walls, then the <u>equivalent image</u> is seen to consist of an infinite "linear string" of point sources (uniformly spaced at a distance apart equal to the width of the rectangular guide). <u>Therefore we may connect the "cylindrical solution" (unknown excitation) to the rectangular guide solution (excitation known from Part II) as follows:</u>

$$\left[\text{SOLUTION FOR POINT SOURCE IN RECTANGULAR GUIDE}\right] = \sum_{\substack{\text{ALL POINT}\\ \text{SOURCE IMAGES}}}\left(\text{SOLUTION FOR EACH POINT SOURCE IN PARALLEL LAYER GUIDE}\right)$$

<u>Since the left hand side is known from Part II, it turns out the "cylindrical mode" excitations, which appear in the right hand side can be determined.</u>

To show the procedure in somewhat more detail, we write the last expression for the top layer as follows:

$$\sum_{\substack{\text{ALL RECTANGULAR}\\ \text{GUIDE MODES}}}\left(\text{VELOCITY POTENTIAL OF EACH RECTANGULAR GUIDE MODE}\right) = \sum_{\substack{\text{ALL "CYLINDRICAL"}\\ \text{MODES (INDEX } m)}}\left\{\sum_{\substack{\text{ALL IMAGES}\\ (\text{INDEX } j)}} A_{1m} J_0(k_m r_j)\sin\beta_{1m} z\right\}$$

Upon evaluating the inner summation on the right hand side it is found that the (remaining) summand may be equated to the summand on the left hand side, so that the A_{lm} (the "excitations") may be obtained.

10. Additional Comments

a) The same comments appearing in paragraphs 8a and 8b of Part II apply here.

RESULTS AND CONCLUSIONS

The problem of the acoustic wave produced in N lossless plane parallel liquid layers due to point source excitation has been solved for the far field in a form permitting of practical computation. The solution is expressed as a sum of normal modes: This constitutes a generalization of the results of Pekeris who obtained solutions in the normal mode form for the special cases of two and three layers (Reference (a)).

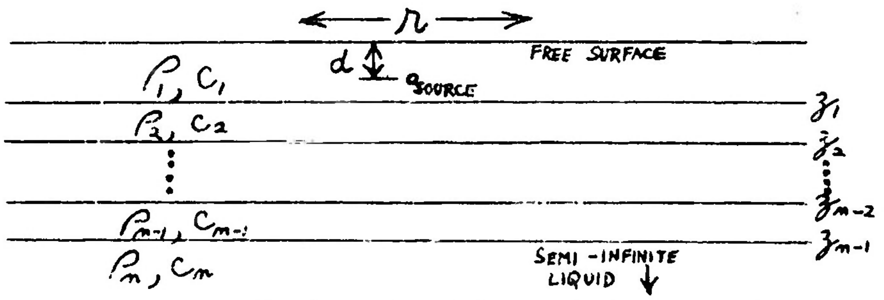
The method employed here for obtaining the mode excitations was basically different from the usual techniques of finding Green's functions. This method greatly simplified the present problem, and may similarly be expected to facilitate the procedure in various other excitation problems.

Appendix A: Computation Procedure

For practical computing purposes, as well as for the sake of conciseness, the results will now be summarized in a more directly applicable form:

1. Statement of Problem

To obtain the far field velocity potential produced in a system of n parallel dissipationless liquid layers by a point source of sound oscillating at a frequency ω . The system is shown in the following diagram.



A necessary condition for the existence of propagating modes is that the wave velocities c_j must not be monatonically increasing with successively higher layers.

2. Form of the Far Field Solution

$$\phi_{jm\,(\text{outgoing wave, far field})} = \xi_{jm}\left(\frac{1}{\sqrt{2\pi k_m r}}\right)\cos\left(\omega t - k_m r - \frac{\pi}{4}\right)$$

Where the j subscript refers to the layer (as used for the ρ's and c's in the above diagram) and the m subscript to the mode.

And where

$$\left\{\begin{array}{ll}
\xi_{jm} \equiv A_{jm}\sin\beta_{jm}z + B_{jm}\cos\beta_{jm}z & \text{FOR } j = 2, 3, \cdots m-1. \\
\xi_{1m} \equiv A_{1m}\sin\beta_{1m}z & \\
\xi_{mm} \equiv A_{mm}e^{-i\beta_{mm}z} & \\
\beta_{jm} \equiv \sqrt{\frac{\omega^2}{c_j^2} - k_m^2} \equiv k_m\sqrt{\frac{c_m^2}{c_j^2} - 1} & \text{FOR } j \neq m \\
\beta_{mm} \equiv -k\sqrt{\frac{c_m^2}{c_j^2} - 1} \quad (\text{FOR } j = m) & \\
c_m \equiv [\text{PHASE VELOCITY OF } m\text{'th MODE}] & (c_m \text{ THE SAME IN ALL LAYERS})
\end{array}\right.$$

3. <u>Finding the k's of the Propagating Modes, and the Linear Relations Between the A_{jm} and B_{jm} of Any Given Mode.</u>

First obtain the β_j as functions of k (eigenvalues of k unknown as yet) and the c_j using the relations given in the preceding paragraph for the β_{jm}: Here the m subscripts are omitted because the β_j are regarded temporarily as continuous functions of k, rather than as discrete values corresponding to modes.

Then apply Eqns. (1) and (2) of Part II to the lowest boundary and Eqns. (3) and (4) to each successively higher boundary through z_1. This gives the A_j and B_j <u>as functions of k and A_m.</u>

Next we must find the k values for which $B_1(k, A_m) = 0$. These k values correspond to the propagating modes, and are designated by a subscript as k_m. The corresponding β_j, A_j and B_j values are thus designated β_{jm}, A_{jm}, B_{jm} respectively. If three or more layers are involved, the k values for which $B_1 = 0$ are in general best found by graphing B_1 against k. (The unknown A_m value has no effect upon the roots.) Only real values of k which satisfy this condition correspond to propagating modes.

4. <u>Finding the Coefficient A_{1m}; And Consequently Obtaining All The A_{jm} and B_{jm} of The Mode.</u>

The coefficient A_{1m} is given by Eqn. (11C). The other A_{jm} and B_{jm} of the mode may then be determined by virtue of their relations to A_{1m} through Eqns. (1), (2), (3), (4). (Eqns. (1), (2), (3), (4) are such as to yield readily the A_{jm} and B_{jm} in terms of the bottom coefficient A_{mm} but must be essentially "inverted" in order to give the A_{jm} and B_{jm} in terms of A_{1m} instead. This inversion of the equations is a very simple iterative process and will not be done here.)

5. <u>Far Field m'th Mode Solution for Outgoing Wave Due to a Point Source in the Top Layer.</u>

Having obtained all the β_{jm}, A_{jm}, and B_{jm} corresponding to each k_m we have the solution for the velocity potential of the m'th mode in the j'th layer by substituting into the expression for $\phi_{jm\,(\text{OUTGOING WAVE, FAR FIELD})}$ given in paragraph 2 above.

Appendix B: Comparison of Two Layer Case to Results of Reference (a).

1. For the two layer case (i.e. - a single layer over a semi-infinite region) Eqns. (1) and (2) of Part I become:

$$A_1 = A_2 e^{-i\beta_2 z_1}\left[\frac{\rho_2}{\rho_1}\sin\beta_1 z_1 - \frac{i\beta_2}{\beta_1}\cos\beta_1 z_1\right] \qquad \text{EQN (1A)}$$

$$B_1 = 0 = A_2 e^{-i\beta_2 z_1}\left[\frac{\rho_2}{\rho_1}\cos\beta_1 z_1 + \frac{i\beta_2}{\beta_1}\sin\beta_1 z_1\right] \qquad \text{EQN (2A)}$$

From Eqn. (2A) we have

$$\frac{\rho_2}{\rho_1}\cos\beta_1 z_1 = -\frac{i\beta_2}{\beta_1}\sin\beta_1 z_1$$

or

$$\tan\beta_1 z_1 = i\frac{\rho_2}{\rho_1}\frac{\beta_1}{\beta_2} \qquad \text{EQN (2B)}$$

If the β's in this last equation are expressed in terms of k, those values of k satisfying the relation correspond to the propagating modes. If expressed in the appropriate notation this equation becomes identical with Eqn. (A49) of reference (a).

2. Eqn. (11C) of Part III becomes for the two layer case (omitting the m subscripts will not cause confusion):

$$A_1 = \frac{\rho_1\omega\left(a_1\sin\beta_1 d\right)^2\cdot\delta V}{2\sin\beta_1 d\cdot\left[\rho_1\int_0^{z_1} a_1^2\sin^2\beta_1 z\,dz + \rho_2\int_{z_1}^{\infty} a_2^2 e^{-2i\beta_2 z}\,dz\right]}$$

Where a_1 and a_2 are coefficients corresponding to a "mathematically fictional" m'th mode of arbitrary strength. That is the a_1 and a_2 obey Eqns. (1A) and (2A). The expression readily becomes:

$$A_1 = \frac{\omega\,\delta V\sin\beta_1 d}{z_1 - \frac{\sin 2\beta_1 z_1}{2\beta_1} + \frac{1}{i\beta_2}\frac{\rho_2}{\rho_1}\frac{a_2^2}{a_1^2}e^{-2i\beta_2 z_1}}$$

Since a_1 and a_2 satisfy Eqns. (1A) and (2A), we may write Eqn. (_A) as:

$$\frac{a_2}{a_1} = \frac{A_2}{A_1} = \frac{1}{e^{-i\beta_2 z_1}\left[\frac{\rho_2}{\rho_1}\sin\beta_1 z_1 - \frac{i\beta_2}{\beta_1}\cos\beta_1 z_1\right]}$$

or

$$\frac{a_2^2}{a_1^2} = \frac{1}{e^{-2i\beta_2 z_1}\left[\frac{\rho_2}{\rho_1}\sin\beta_1 z_1 - \frac{i\beta_2}{\beta_1}\cos\beta_1 z_1\right]^2}$$

hence

$$A_1 = \frac{\omega\,\delta V \sin\beta_1 d}{z_1 - \frac{\sin 2\beta_1 z_1}{2\beta_1} + \frac{1}{i\beta_2}\frac{\rho_2}{\rho_1}\left(\frac{1}{\frac{\rho_2}{\rho_1}\sin\beta_1 z_1 - i\frac{\beta_2}{\beta_1}\cos\beta_1 z_1}\right)^2}$$

Introducing the notation used by Pekeris (Reference (a)), we have:

$$\beta_1 z_1 \equiv \chi_m \quad ; \quad \frac{\rho_1}{\rho_2} \equiv b \quad ; \quad z_1 \equiv H$$

Using Eqn. (2B) above, we get, in this notation:

$$A_1 = \frac{\omega\,\delta V \sin\beta_1 d}{H - \frac{\sin 2\chi_m}{2\beta_1} - \frac{\tan\chi_m}{\beta_1}\left(\frac{b}{\sin\chi_m + \frac{\cos\chi_m}{\tan\chi_m}}\right)^2}$$

which reduces to:

$$A_1 = \frac{\omega\left(\frac{\chi_m}{H}\right)(\delta V)\sin\beta_1 d}{\chi_m - \sin\chi_m\cos\chi_m - b^2\tan\chi_m\sin^2\chi_m}$$

Comparing to the definition of $F(\chi_n)$ given in Eqn. (A73) of Reference (a), we may write:

$$A_1 = \frac{\omega\,\delta V}{H}\,(\sin\beta_1 d)\cdot F(\chi_n)$$

Therefore the solution as given in paragraph 2 of Appendix A becomes, for the top layer:

$$\phi_{1m\,(\text{OUTGOING WAVE, FAR FIELD})} = \left[\frac{\omega\,\delta V}{H}(\sin\beta_{1m} d)\cdot F(\chi_n)\right](\sin\beta_{1m} z)\left(\frac{1}{\sqrt{2\pi k_m r}}\right)\cos\left(\omega t - k_m r - \frac{\pi}{4}\right)$$

Putting this into complex form, and writing $\beta_1 \equiv \frac{\chi_n}{z_1} \equiv \frac{\chi_n}{H}$, this becoms:

$$\boxed{\phi_{1m\,(\text{OUTGOING WAVE, FAR FIELD})} = \left(\frac{\delta V}{2}\right)\left(\frac{\omega}{H}\right)\left(\sqrt{\frac{2}{\pi r}}\right)\frac{1}{\sqrt{k}}\,e^{i(\omega t - kr - \frac{\pi}{4})}\,F(\chi_n)\sin\left(\frac{\chi_n d}{H}\right)\cdot\sin\left(\frac{\chi_n z}{H}\right)}$$

If the source strength δV is put equal to $(\frac{4\pi}{\omega})$ this solution becomes identical with Pekeris' result (Eqn. A71).

3. The question of whether the present solution is identical with that of Pekeris depends upon whether or not he chose the source strength as $\frac{4\pi}{\omega}$, as is required by the above comparison of results. At first sight the two results would appear contradictory since $\frac{4\pi}{\omega}$ does not have the dimensions of volume appropriate to a source displacement. However there is actually no disagreement, for it will be shown that Pekeris nevertheless did in effect (i.e. - implicitly) choose the source strength as $\frac{4\pi}{\omega}$: Since there was no particularly important reason to do so at the time, he did not attempt to keep his equations dimensionally consistent; and by introducing the source strength indirectly through a boundary condition, an implicit consequence was that the source strength was $\frac{4\pi}{\omega}$. To show this we proceed as follows:

We will make use of equations (A17) and (A18) of Reference (a). We notice that each of these is dimensionally inconsistent, its right hand side having the dimensions of length while the left hand side has the dimensions of a velocity potential $(\frac{L^2}{T})$. The discontinuity in the particle velocity at the plane $z=d$ (which is a plane containing the source) has a vertical component given by $e^{i\omega t}\left(\frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_2}{\partial z}\right)$.

Using the values of ψ_1 and Γ_1 given by equations (A17) and (A18) this discontinuity is readily shown to be equal to $2e^{i\omega t}\int_0^\infty J_0(kr)\,k\,dk$. This is a discontinuous integral, being infinite at $r=0$ and zero everywhere else. However, if this discontinuous integral is integrated over the $z=d$ plane the result is finite. If this multiple integration is carried out (for example with the help of equation (6), pg. 386 of Watson's "Bessel Functions", 2nd Edition) the value is found to be $4\pi e^{i\omega t}$. Knowing the integrated value of the vertical velocity discontinuity over the plane $z=d$, we can, by simple flow considerations, show that the maximum volume displacement of the source is given by $\delta V = \frac{8\pi}{\omega}$. However equations (A17) and (A18) represent the standing wave solution. If only the outgoing wave part of the solution is kept it can be shown, as may be expected, that the source strength becomes halved: $\delta V = \frac{4\pi}{\omega}$. But this is the value that was required to verify the equivalence between the present solution and that of Pekeris.

4. Thus, we may conclude that the present results are in agreement with those of Pekeris.

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